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Quantizing Yang–Mills Theory on a 2-Point Space

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Abstract

We perform the Batalin-Vilkovisky quantization of Yang–Mills theory on a 2-point space, discussing the formulation of Connes–Lott as well as Connes' real spectral triple approach. Despite of the model's apparent simplicity the gauge structure reveals infinite reducibility and the gauge fixing is afflicted with the Gribov problem.

1 Introduction

Noncommutative geometry constitutes one of the fascinating new concepts in current theoretical physics research with many promising impacts and applications in a diverse set of fields [1, 2, 3, 4, 5]. Specifically we mention the construction of the classical action of the standard model [6, 7], unifying the Einstein–Hilbert action, the Yang–Mills action, the Dirac action, and the Klein–Gordon action with the Higgs potential and spontaneous symmetry breaking.

The basic idea of noncommutative geometry is to replace the notion of differential manifolds and functions by specific noncommutative algebras of functions; the geometric setting of gauge theories as fibre bundles finds a noncommutative generalization in terms of finitely generated projective modules over noncommutative algebras.

It seems, however, that within this noncommutative algebraic framework the concepts of *quantizing* gauge theories, in specific the issue of gauge fixing and the proper definition of a path integral measure for the standard model are not yet fully understood [8]. Our intention for this paper is not to present new results in these rather fundamental issues. Instead we quantize one of the simplest toy models for noncommutative gauge theories, which is Yang–Mills theory on a 2-point space, by applying the standard Batalin–Vilkovisky method [9, 10, 11, 12]. Somewhat surprisingly we find that despite of the model's original simplicity the gauge structure reveals infinite reducibility and the gauge fixing is afflicted with the Gribov [13] problem.

In section 2 we work out the formulation of the model following the approach of Connes–Lott [14], see also [15]. In section 3 the infinite reducibility of the gauge symmetry is explained; the Batalin–Vilkovisky quantization of the model is performed in section 4. We discuss the Gribov problem in section 5 and finally, in section 6, recast our results within Connes' real spectral triple approach [6, 16].

2 The Formulation of Connes–Lott

Following [14], see also [15], we define the Yang–Mills Theory on a 2-point space in terms of the algebra $\mathbf{A} = C \oplus C$, which is represented by diagonal complex valued 2×2 matrices; the Dirac operator D is given by $D = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$, where $\mu \in \mathrm{R}$ is an arbitrary parameter. The differential p-forms ω_p are constant, diagonal or offdiagonal 2×2 matrices, depending on whether p is even or odd, respectively. One has a $\mathcal{Z}_2$ grading of matrices (to be diagonal or offdiagonal) and obtains a matrix derivative $\mathbf{d}$. Acting on 2×2 matrices it is a nilpotent graded derivation[1] with respect to the matrix product and the matrix $\mathcal{Z}_2$ grading.

Specifically the 1-forms are given by $\omega_1 = a\,\mathbf{d}b$, where $a, b \in \mathbf{A}$, which are odd (i.e. offdiagonal) matrices. The subset of anti-Hermitean 1-forms $\mathcal{A}$ can be parametrized by

$$\mathcal{A} = \begin{pmatrix} 0 & i\mu\phi \\ i\mu\bar{\phi} & 0 \end{pmatrix} \tag{2.1}$$

and constitute the gauge fields of the model; here $\phi \in \mathrm{C}$ denotes a (constant) scalar field. The (rigid) gauge transformations of $\mathcal{A}$ are defined by

$$\mathcal{A}^U = U^{-1}\mathcal{A}U + U^{-1}\mathbf{d}\,U \tag{2.2}$$

with U being a unitary element of the algebra $\mathbf{A}$. It is a constant, even and unitary matrix which we define to have only abelian entries; it can exponentially be parametrized by the even matrix ε

$$U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} = e^{i\varepsilon}, \quad \varepsilon = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \alpha, \beta \in \mathrm{R}. \tag{2.3}$$

We point out that the Yang–Mills theory on the 2-point space is an ideal play ground to study quantization techniques: Due to the nonabelian form of the gauge transformations (2.2) the model shares many interesting features with the standard Yang–Mills theory, yet it has no physical space-time dependence and allows extremely simple calculations.

[1] $\mathbf{d}\,a = i\mu \begin{pmatrix} a_{21} + a_{12} & a_{22} - a_{11} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{pmatrix}$ where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{ij} \in \mathrm{C}$.

We even restrict ourselves to just abelian entries along the diagonal of U, thus studying a $U(1)\times U(1)$ gauge model with nonabelian features.

We define a scalar product for 2×2 matrices a, b by $\langle a\,|\,b\rangle = tr\, a^\dagger\, b$ where $\dagger$ denotes taking the Hermitian conjugate. The curvature $\mathcal{F}$ is defined as usual by $\mathcal{F} = \mathbf{d}\mathcal{A} + \mathcal{A}\,\mathcal{A}$ and transforms under gauge transformations as $\mathcal{F}^U = U^{-1}\mathcal{F}U$; for an action which is automatically invariant under the gauge transformations (2.2) one takes

$$S_{inv} = \frac{1}{2}\langle\mathcal{F}|\mathcal{F}\rangle. \tag{2.4}$$

Written out in components the scalars' contribution is given by

$$S_{inv} = \mu^4\left((\phi+\bar{\phi}) + \phi\,\bar{\phi}\right)^2. \tag{2.5}$$

It was pointed out in [17, 18] that the most general form of the gauge invariant action also allows a term proportional to $tr\mathcal{F}$

$$\hat{S}_{inv} = \frac{1}{2}\Big(\langle\mathcal{F}\,|\,\mathcal{F}\rangle + \gamma\, tr\mathcal{F}\Big) \tag{2.6}$$

where $\gamma\in$ R is an arbitrary parameter. We note, however, that one requires the scalars ϕ to be vanishing at the minimum of the action so that in the case of (2.6) the scalars have to be shifted appropriately. Explicitly we have

$$\hat{S}_{inv} = \mu^4(2u+u^2+v^2)(2u+u^2+v^2-\frac{\gamma}{\mu^2}) \tag{2.7}$$

where we introduced $\phi = u+iv$. Whereas the local maximum is at

$$u_{max} = -1, \quad v_{max} = 0 \tag{2.8}$$

the circle of local minima is given by

$$(u+1)^2+v^2 = 1+\frac{\gamma}{2\mu^2}, \quad \text{where} \quad \gamma\geq -2\mu^2. \tag{2.9}$$

We choose

$$u_{min} = \sqrt{1+\frac{\gamma}{2\mu^2}}-1, \quad v_{min} = 0 \tag{2.10}$$

and define shifted scalars $\tilde{\phi} = \phi - u_{min}$ which by construction are vanishing at the minimum of the action. From

$$\hat{S}_{inv} = \mu^4 \left((\tilde{\phi} + \bar{\tilde{\phi}}) \sqrt{1 + \frac{\gamma}{2\mu^2}} + \tilde{\phi}\bar{\tilde{\phi}} \right)^2 - \frac{\gamma^2}{2}. \tag{2.11}$$

we omit the irrelevant constant $-\frac{\gamma^2}{2}$, rescale $\tilde{\phi} = \hat{\phi}\sqrt{1 + \frac{\gamma}{2\mu^2}}$ and $\mu\sqrt{1 + \frac{\gamma}{2\mu^2}} = \hat{\mu}$ so that finally

$$\hat{S}_{inv} = \hat{\mu}^4 \left((\hat{\phi} + \bar{\hat{\phi}}) + \hat{\phi}\,\bar{\hat{\phi}} \right)^2. \tag{2.12}$$

We see that the inclusion of the action term linear in $\mathcal{F}$ can be compensated by shifting and rescaling of the scalar field ϕ, as well as by rescaling of the parameter μ. As the scalar fields and the parameter are arbitrary from the outset the inclusion of the action term linear in $\mathcal{F}$ appears to be unnecessary. In the following we will set $\mu = 1$ for simplicity and stick to the action term (2.4) quadratic in $\mathcal{F}$.

3 Gauge Transformations and Infinite Reducibility

The (zero-stage) gauge transformations (2.2) explicitly are given by

$$\mathcal{A}^U = \begin{pmatrix} 0 & ie^{i(\beta-\alpha)}(\phi+1) - i \\ ie^{-i(\beta-\alpha)}(\bar{\phi}+1) - i & 0 \end{pmatrix}, \tag{3.1}$$

so that the usual abelian gauge transformations are implied for the Higgs fields $H = \phi + 1$ and $\bar{H} = \bar{\phi} + 1$. To discuss infinitesimal (zero-stage) gauge transformations we introduce an even, infinitesimal (zero stage) gauge parameter matrix ε_e^0 in terms of which $U \simeq \mathbf{1} + \varepsilon_e^0$. The infinitesimal (zero-stage) gauge variation of $\mathcal{A}$ derives as

$$\delta_{\varepsilon_e^0}\mathcal{A} = i\mathbf{R}^0\,\varepsilon_e^0 \quad \text{where} \quad \mathbf{R}^0 = \mathbf{D}; \tag{3.2}$$

here the (zero-stage) gauge generator $\mathbf{R}^0$ is defined in terms of the covariant matrix derivative $\mathbf{D}$, which acting on ε_e^0 is given by $\mathbf{D}\varepsilon_e^0 = \mathbf{d}\varepsilon_e^0 + [\mathcal{A}, \varepsilon_e^0]$.

A gauge symmetry is called irreducible if the (zero stage) gauge generator $\mathbf{R}^0$ does not posess any zero mode [9, 10, 11, 12].

It is amusing to note that the Yang–Mills theory on the 2-point space reveals an infinitely reducible gauge symmetry: We observe that $\mathbf{D}\,\mathbf{d}$ is vanishing on arbitrary odd matrices. Thus there exists a zero mode ε_e^1 for the (zero-stage) gauge generator $\mathbf{R}^0$, such that

$$\mathbf{R}^0\varepsilon_e^1 = 0 \quad \text{where} \quad \varepsilon_e^1 = \mathbf{R}^1\varepsilon_o^1 \quad \text{with} \quad \mathbf{R}^1 = \mathbf{d}. \tag{3.3}$$

Here ε_o^1 denotes an odd, infinitesimal (first-stage) gauge parameter matrix and $\mathbf{R}^1$ the corresponding (first-stage) gauge generator. As a matter of fact an infinite tower of (higher-stage) gauge generators $\mathbf{R}^s$, $s = 1, 2, 3, \cdots$ with never ending gauge invariances for gauge invariances is arising: We define $\mathbf{R}^s = \mathbf{d}$ for $s = 1, 2, 3, \cdots$ so that for each gauge generator there exists an additional zero mode

$$\begin{array}{lll} \mathbf{R}^1\varepsilon_o^2 = 0, & \text{where} & \varepsilon_o^2 = \mathbf{R}^2\varepsilon_e^2 \\ \mathbf{R}^2\varepsilon_e^3 = 0, & \text{where} & \varepsilon_e^3 = \mathbf{R}^3\varepsilon_o^3 \\ \quad\cdots & & \quad\cdots \end{array} \tag{3.4}$$

due to the nilpotency $\mathbf{d}^2 = 0$.

4 Gauge Fixing and BV-Quantization

In this section we straightforwardly apply the usual field theory BV-path integral quantization scheme [10, 11, 12] to the Connes–Lott 2-point model: In addition to the original gauge field $\mathcal{A}$, which for notational convenience we temporarily denote by $\mathcal{A} \equiv \mathcal{C}_{-1}^{-1}$, we introduce ghost fields $\mathcal{C}_s^k$, $\infty \geq s \geq -1$, $s \geq k \geq -1$ with k odd, as well as auxiliary ghost fields $\bar{\mathcal{C}}_s^k$, $\infty \geq s \geq 0$, $s \geq k \geq 0$ with k even (see Fig. 1).

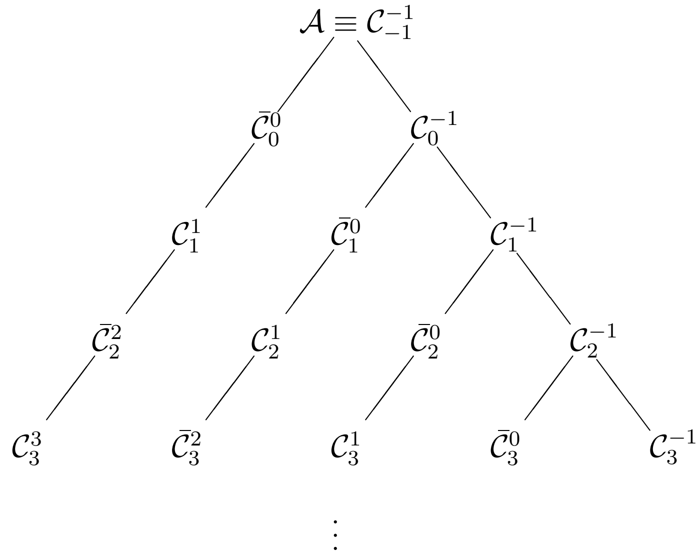

Figure 1. The Infinite Tower of Ghost Fields

Furthermore we add Lagrange multiplier fields π^k_s, $\infty \geq s \geq 1$, $s \geq k \geq 1$ with k odd and $\bar{\pi}^k_s$, $\infty \geq s \geq 0$, $s \geq k \geq 0$ with k even. Finally we introduce antifields $\mathcal{C}^{k*}_s$, $\bar{\mathcal{C}}^{k\,*}_s$. All the ghosts $\mathcal{C}^k_s$, $\bar{\mathcal{C}}^k_s$, multiplier fields π^k_s, $\bar{\pi}^k_s$ and antifields $\mathcal{C}^{k*}_s$, $\bar{\mathcal{C}}^{k\,*}_s$ are matrices which are even for s even and odd for s odd, respectively. We define all the ghost fields $\mathcal{C}^k_s$, $\bar{\mathcal{C}}^k_s$ to be anti-Hermitean, all the multiplier fields π^k_s, $\bar{\pi}^k_s$ to be Hermitean. When s is taken to be odd the ghosts are bosonic whereas the multiplier fields are fermionic; for s even the ghosts are fermionic and the multiplier fields are bosonic, respectively.

An important quantity for the construction of the BV-action is the commutator of (zero-stage) infinitesimal gauge transformations $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\mathcal{A}$, where $\delta_{\varepsilon_k}\mathcal{A} = i\mathbf{R}^0\,\varepsilon_k$ with even matrices ε_k, $k = 1, 2$. It is easy to see that this commutator is vanishing. The BV-action therefore obtains as

$$S_{BV} = S_{inv} + S_{aux} - \langle \mathcal{C}^{-1*}_{-1} | \mathbf{D}\, \mathcal{C}^{-1}_0 \rangle - \sum_{s=1,3,5,\cdots}^{\infty} \langle \mathcal{C}^{-1*}_s | \mathbf{d}\, \mathcal{C}^{-1}_{s+1} \rangle - i \sum_{s=0,2,4,\cdots}^{\infty} \langle \mathcal{C}^{-1*}_s | \mathbf{d}\, \mathcal{C}^{-1}_{s+1} \rangle, \quad (4.1)$$

where we denote by S_{aux} the auxiliary field action

$$S_{aux} = \sum_{k=0,2,4,\cdots}^{\infty} \sum_{s=k}^{\infty} \langle \bar{\pi}^k_s | \bar{\mathcal{C}}^{k\,*}_s \rangle + \sum_{k=1,3,5,\cdots}^{\infty} \sum_{s=k}^{\infty} \langle \mathcal{C}^{k*}_s | \pi^k_s \rangle. \quad (4.2)$$

By $\boldsymbol{\delta}$ we denote a nilpotent matrix coderivative operator[2] which is defined by $\langle \boldsymbol{\delta} a_o | b_e\rangle = \langle a_o | \mathbf{d} b_e\rangle$ and $\langle \boldsymbol{\delta} a_e | b_o\rangle = \langle a_e | \mathbf{d} b_o\rangle$. It allows to define gauge fixing conditions

$$
\begin{aligned}
\boldsymbol{\delta}\mathcal{C}_s^k &= 0, \qquad \infty \geq s \geq -1, \quad s \geq k \geq -1 \quad \text{with}\, k \quad \text{odd} \\
\boldsymbol{\delta}\bar{\mathcal{C}}_s^k &= 0, \qquad \infty \geq s \geq 0, \quad s \geq k \geq 0 \quad \text{with } k \quad \text{even},
\end{aligned} \tag{4.3}
$$

which are similar to the Feynman gauge in standard Yang–Mills theory. In the BV-approach we implement these gauge fixing conditions by defining the gauge fixing fermion $\Psi = \Psi_\delta + \Psi_\pi$ by

$$
\begin{aligned}
\Psi_\delta &= \sum_{s=0,2,4,\cdots}^{\infty} \sum_{k=0,2,4,\cdots\; k\leq s} \Big(-\langle \bar{\mathcal{C}}_s^k \,|\, \boldsymbol{\delta}\mathcal{C}_{s-1}^{k-1}\,\rangle + \langle \boldsymbol{\delta}\bar{\mathcal{C}}_{s+1}^k \,|\, \mathcal{C}_{s+2}^{k+1}\,\rangle \\
&\qquad + i\langle \bar{\mathcal{C}}_{s+1}^k \,|\, \boldsymbol{\delta}\mathcal{C}_s^{k-1}\,\rangle + i\langle \boldsymbol{\delta}\bar{\mathcal{C}}_s^k \,|\, \mathcal{C}_{s+1}^{k+1}\,\rangle \Big), \\
\Psi_\pi &= \frac{1}{2} \sum_{s=0,2,4,\cdots}^{\infty} \sum_{k=0,2,4,\cdots\; k<s} \Big(\langle \bar{\mathcal{C}}_s^k \,|\, \pi_s^{k+1}\,\rangle + \langle \bar{\pi}_s^k \,|\, \mathcal{C}_s^{k+1}\,\rangle \\
&\qquad + i\langle \bar{\mathcal{C}}_{s+1}^k \,|\, \pi_{s+1}^{k+1}\,\rangle + i\langle \bar{\pi}_{s+1}^k \,|\, \mathcal{C}_{s+1}^{k+1}\,\rangle \Big) \\
&\quad + \frac{1}{2} \sum_{k=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_k^k \,|\, \bar{\pi}_k^k\,\rangle.
\end{aligned} \tag{4.4}
$$

We eliminate the antifields by using the gauge fixing fermion Ψ via

$$
\langle \mathcal{C}_s^{k*}| = \frac{\partial \Psi}{\partial |\mathcal{C}_s^k\rangle}, \qquad |\bar{\mathcal{C}}_s^{k\,*}\rangle = \frac{\partial \Psi}{\partial \langle \bar{\mathcal{C}}_s^k|}, \tag{4.5}
$$

so that the gauge fixed action S_Ψ reads

$$
\begin{aligned}
S_\Psi &= S_{inv} - i\langle \bar{\mathcal{C}}_0^0 \,|\, \boldsymbol{\delta}\mathbf{D}\,\mathcal{C}_0^{-1}\rangle - i \sum_{s=1,3,5,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, \boldsymbol{\delta}\mathbf{d}\,\mathcal{C}_{s+1}^{-1}\rangle + \sum_{s=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, \boldsymbol{\delta}\mathbf{d}\,\mathcal{C}_{s+1}^{-1}\rangle \\
&\quad + \sum_{k=0,2,4,\cdots}^{\infty} \sum_{s=k+1,\,odd}^{\infty} \Big(i\langle \bar{\pi}_s^k \,|\, \pi_s^{k+1}\rangle + \langle \bar{\pi}_s^k \,|\, (i\boldsymbol{\delta}\mathcal{C}_{s-1}^{k-1} + \mathbf{d}\mathcal{C}_{s+1}^{k+1})\rangle \\
&\qquad + \langle (i\boldsymbol{\delta}\bar{\mathcal{C}}_{s-1}^k - \mathbf{d}\bar{\mathcal{C}}_{s+1}^{k+2}) |\, \pi_s^{k+1}\rangle \Big) \\
&\quad + \sum_{k=0,2,4,\cdots}^{\infty} \sum_{s=k+2,\,even}^{\infty} \Big(\langle \bar{\pi}_s^k \,|\, \pi_s^{k+1}\rangle + \langle \bar{\pi}_s^k \,|\, (-\boldsymbol{\delta}\mathcal{C}_{s-1}^{k-1} + i\mathbf{d}\mathcal{C}_{s+1}^{k+1})\rangle \\
&\qquad + \langle (\boldsymbol{\delta}\bar{\mathcal{C}}_{s-1}^k + i\mathbf{d}\bar{\mathcal{C}}_{s+1}^{k+2}) |\, \pi_s^{k+1}\rangle \Big) \\
&\quad + \sum_{k=0,2,4,\cdots}^{\infty} \langle \bar{\pi}_k^k \,|\, (-\boldsymbol{\delta}\mathcal{C}_{k-1}^{k-1} + i\mathbf{d}\mathcal{C}_{k+1}^{k+1} + \frac{1}{2}\bar{\pi}_k^k)\rangle.
\end{aligned} \tag{4.6}
$$

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[2] $\boldsymbol{\delta}\, a = i \begin{pmatrix} a_{12} - a_{21} & -a_{11} - a_{22} \\ -a_{11} - a_{22} & -a_{12} + a_{21} \end{pmatrix}$ where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{ij} \in \mathrm{C}$.

We can now eliminate the Lagrange multiplier fields π_s^k and $\bar{\pi}_s^k$ and arrive at

$$
\begin{aligned}
S_\Psi \longrightarrow \quad & S_{inv} + \frac{1}{2}\langle \mathcal{A} \,|\, \boldsymbol{\delta}\mathbf{d}\,\mathcal{A}\rangle - i\langle \bar{\mathcal{C}}_0^0 \,|\, (\boldsymbol{\delta}\mathbf{D} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_0^{-1}\rangle \\
& - i \sum_{s=1,3,5,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_{s+1}^{-1}\rangle \\
& + \sum_{s=0,2,4,\cdots}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^0 \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_{s+1}^{-1}\rangle \\
& - i \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+1,\,odd}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^{k+2} \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_{s+1}^{k+1}\rangle \\
& + \sum_{k=0,2,4,\cdots}^{\infty} \; \sum_{s=k+2,\,even}^{\infty} \langle \bar{\mathcal{C}}_{s+1}^{k+2} \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_{s+1}^{k+1}\rangle \\
& + \frac{1}{2} \sum_{k=0,2,4,\cdots}^{\infty} \langle \mathcal{C}_{k+1}^{k+1} \,|\, (\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_{k+1}^{k+1}\rangle . \qquad (4.7)
\end{aligned}
$$

All the higher-stage ghost contributions can be integrated away without any effect as $\boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta} = 4 \cdot \mathbf{1}$ and we simply obtain

$$
S_\Psi \longrightarrow S_{inv} + \frac{1}{2}\langle \mathcal{A} \,|\, \boldsymbol{\delta}\mathbf{d}\,\mathcal{A}\rangle - i\langle \bar{\mathcal{C}}_0^0 \,|\, (\boldsymbol{\delta}\mathbf{D} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_0^{-1}\rangle \qquad (4.8)
$$

We see that the gauge fixed action contains the invertible quadratic part $2\langle \mathcal{A} \,|\, \mathcal{A}\rangle$ for the gauge field, as well as $-4i\langle \bar{\mathcal{C}}_0^0 \,|\, \mathcal{C}_0^{-1}\rangle$ for the $\bar{\mathcal{C}}_0^0$, $\mathcal{C}_0^{-1}$ ghost fields.

5 The Gribov Problem

The Yang–Mills theory on the 2-point space suffers from a Gribov problem [13] even for the abelian $U(1) \times U(1)$ case. This can be demonstrated easily by recasting the ghost part of the gauge fixed action (4.8) into the form

$$
\langle \bar{\mathcal{C}}_0^0 \,|\, (\boldsymbol{\delta}\mathbf{D} + \mathbf{d}\boldsymbol{\delta})\,\mathcal{C}_0^{-1}\rangle = \begin{pmatrix} \bar{c}_1 & \bar{c}_2 \end{pmatrix} \begin{pmatrix} 4 + \phi + \bar{\phi} & -\phi - \bar{\phi} \\ -\phi - \bar{\phi} & 4 + \phi + \bar{\phi} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad (5.1)
$$

where we introduced the component ghosts fields $\bar{c}_1$, $\bar{c}_2$ and c_1, c_2 which are the diagonal elements of $\bar{\mathcal{C}}_0^0$ and $\mathcal{C}_0^{-1}$, respectively. The Faddeev–Popov matrix M_{FP}

$$
M_{FP} = \begin{pmatrix} 4 + \phi + \bar{\phi} & -\phi - \bar{\phi} \\ -\phi - \bar{\phi} & 4 + \phi + \bar{\phi} \end{pmatrix} \qquad (5.2)
$$

has a vanishing determinant for $2 + \phi + \bar{\phi} = 0$ which forces $\phi = u + iv$ to lie on the line $u = -1$. We note the distinguished value $\phi = -1$, which we discussed previously by demanding the action to be maximal, see (2.8). Now this value arises by inserting the gauge fixing condition $\boldsymbol{\delta}\,\mathcal{A} = 0$ into the Faddeev–Popov determinant $det\,M_{FP}$.

We observe that the classical action S_{inv} not only has an invariance under the (rigid) gauge transformations (2.2), but also under the discrete charge conjugation operation (conveniently expressed in terms of the Higgs fields H, $\bar{H}$)

$$H \longrightarrow -\bar{H}, \qquad \bar{H} \longrightarrow -H. \tag{5.3}$$

After the spontaneous symmetry breakdown this discrete symmetry guarantees that the minima of the action are degenerated. In the quantum case, however, due to the Gribov problem, these discrete jumps no longer are allowed and the quantum corrections to the action will lift the classical degeneracy of the minima.

6 Connes' Real Spectral Triple Formulation

The formulation of the Yang–Mills theory model on the 2-point space in terms of Connes's real spectral triple approach proceeds by specifying the spectral triple $(\mathbf{A}, \mathcal{H}, D)$ together with the antilinear isometry $\mathcal{J}$, fulfilling a set of specific properties [6, 16]. We represent the elements $a = (a_1, a_2, a_3)$ of the algebra $\mathbf{A} = C \oplus C \oplus C$, as well as D and $\mathcal{J}$, by specific 4×4 matrices; the Hilbert space $\mathcal{H}$ simply is C^4. Specifically we have

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \circ c.c. \tag{6.1}$$

where c.c. denotes complex conjugation. As an example one sees that for $a,\, b \in \mathbf{A}$ the differential 1-form $\omega_1 = a\,[i\,D, b]$ is given by

$$\omega_1 = i \begin{pmatrix} 0 & a_1(b_2 - b_1) & 0 & 0 \\ a_2(b_1 - b_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.2}$$

We recognize that apart of irrelevant zeros in the upper right, lower left and lower right matrix corners of the differential forms our previous discussion of the gauge symmetries, the gauge fixing and the ghost structure proves right as well.

We conclude that the quantization of the Yang–Mills theory model on the 2-point space within the Connes–Lott scheme and within Connes' real spectral triple approach are equivalent; the model reveals infinite reducibility and is afflicted with the Gribov problem.

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